Behavior of $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$
To accompany Cauchy horizons in Gowdy space times
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I. $v(\theta)^2 \neq 1$

Starting with

$$P(t,\theta) = P_\infty(\theta) - v(\theta)\ln(t) + V(\theta)t^2(\ln(t))^2 + V_1(\theta)t^2 + V_2(\theta)t^2\ln(t) \quad (1)$$

and

$$Q(t,\theta) = Q_\infty(\theta) + q(\theta)t^{2\,v(\theta)} + qq(\theta)t^{2\,v(\theta)}\ln(t), \quad (2)$$

it follows that

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} \to 1/4\,\frac{e^{\gamma(t,\theta)}((v(\theta))^2+3)(v(\theta)-1)^2(v(\theta)+1)^2}{t^3} \quad (3)$$

as $t \to 0$. See limitone.mws. This distinguishes the case $v(\theta)^2 = 1$. For $v(\theta)^2 \neq 1$ with $e^{\gamma(t,\theta)} \sim 1/t$ then $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ diverges like $1/t^4$.

Note: Instead of (1) and (2) one can use (7) and (8) given below. One again recovers (3) explicitly (see limitonerevised.mws). Due to the growth of terms this calculation takes about 45 times as long to do with (7) and (8) than it does with (1) and (2).

II. $v(\theta) = 1,\ \frac{dv(\theta)}{d\theta} \neq 0$

With (1) and (2), subject to the consistency condition (see constraintvone.mws)

$$\frac{d}{d\theta}Q_\infty(\theta) = 0, \quad (4)$$

but keeping $\frac{d^nQ_\infty(\theta)}{d\theta^n}$, $n = 2,3,4$ and $\frac{d^nv(\theta)}{d\theta^n}$, $n = 2,3,4$ it follows that

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} \to 4\,\frac{(-(\frac{d}{d\theta}v(\theta))^2 + e^{2\,P_\infty(\theta)}(\frac{d^2}{d\theta^2}Q_\infty(\theta))^2)e^{\gamma(t,\theta)}}{t} \quad (5)$$

as $t \to 0$. See limittwo.mws. For $v(\theta) = 1$ and $\frac{dv(\theta)}{d\theta} \neq 0$ with $(\frac{d}{d\theta}v(\theta))^2 \neq e^{2\,P_\infty(\theta)}(\frac{d^2}{d\theta^2}Q_\infty(\theta))^2$ and $e^{\gamma(t,\theta)} \sim 1/t$ then $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ diverges only like $1/t^2$.

Note: Instead of (1) and (2) one can use (7) and (8) given below. One again recovers (5) explicitly (see limittworevised.mws). Again due to the growth of terms this calculation takes longer, but with the simplification (4) only about 4 times as long to do with (7) and (8) than it does with (1) and (2).

III. $v(\theta) = 1,\ \frac{dv(\theta)}{d\theta} \neq 0,\ (\frac{d}{d\theta}v(\theta))^2 = e^{2\,P_\infty(\theta)}(\frac{d^2}{d\theta^2}Q_\infty(\theta))^2$

For the special case $(\frac{d}{d\theta}v(\theta))^2 = e^{2\,P_\infty(\theta)}(\frac{d^2}{d\theta^2}Q_\infty(\theta))^2$, in the same manner as (5) we get

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} \to 3\,t\,(\ln(t))^4\,e^{\gamma(t,\theta)}\left(\frac{d^2}{d\theta^2}Q_\infty(\theta)\right)^4\left(e^{P_\infty(\theta)}\right)^4 \quad (6)$$

so that with $e^{\gamma(t,\theta)} \sim 1/t$ then $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ now diverges like $(\ln(t))^4$. See limittwospecial.mws.

IV. $v(\theta)=1$, $\frac{dv(\theta)}{d\theta}=0$

For $v(\theta)=1$ and $\frac{dv(\theta)}{d\theta}=0$ it is necessary to use the more explicit forms than (1) and (2) because relations amongst the functions used in (1) and (2) lead to a degeneracy (see limitthree.mws). With

$$P(t,\theta)=P_\infty(\theta)-v(\theta)\ln(t)+1/4\,(e^{P_\infty(\theta)})^2(\frac{d^2}{d\theta^2}Q_\infty(\theta))^2t^2(\ln(t))^2 \tag{7}$$
$$+V_1(\theta)t^2+(e^{2\,P_\infty(\theta)}\psi_Q(\theta)\frac{d^2}{d\theta^2}Q_\infty(\theta)-1/4\,(\frac{d^2}{d\theta^2}Q_\infty(\theta))^2-1/4\,\frac{d^2}{d\theta^2}v(\theta))t^2\ln(t)$$

and

$$Q(t,\theta)=Q_\infty(\theta)+\psi_Q(\theta)t^{2\,v(\theta)}+1/2\,(\frac{d^2}{d\theta^2}Q_\infty(\theta))t^{2\,v(\theta)}\ln(t), \tag{8}$$

and with the consistency condition (4),

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\to 4\,\frac{e^{2\,P_\infty(\theta)}(\frac{d^2}{d\theta^2}Q_\infty(\theta))^2e^{\gamma(t,\theta)}}{t} \tag{9}$$

as $t\to 0$ where it is assumed that $\frac{d^2Q_\infty(\theta)}{d\theta^2}\neq 0$. See limitfour.mws. This agrees with limittwo.mws.

V. $v(\theta)=1$, $\frac{dv(\theta)}{d\theta}=\frac{d^2Q_\infty(\theta)}{d\theta^2}=0$

For the special case $\frac{d^2Q_\infty(\theta)}{d\theta^2}=0$ we obtain (see limitfive.mws)

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\to 3\,((\frac{d^2}{d\theta^2}v(\theta))^2-(\frac{d^3}{d\theta^3}Q_\infty(\theta))^2e^{2\,P_\infty(\theta)})t(\ln(t))^2e^{\gamma(t,\theta)} \tag{10}$$

as $t\to 0$ so that with $e^{\gamma(t,\theta)}\sim 1/t$ then $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ diverges only like $(\ln(t))^2$. The special cases $\frac{d^2}{d\theta^2}v(\theta)=0$ or $\frac{d^3}{d\theta^3}Q_\infty(\theta))^2=0$ are shown to be consistent with (10) in limitfivev2.mws and limitfiveq3.mws

VI. $v(\theta)=1$, $\frac{dv(\theta)}{d\theta}=\frac{d^2Q_\infty(\theta)}{d\theta^2}=0$, $(\frac{d^2}{d\theta^2}v(\theta))^2=(\frac{d^3}{d\theta^3}Q_\infty(\theta))^2e^{2\,P_\infty(\theta)}\neq 0$

As shown in limitfivespecial.mws, under the above conditions

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\to -3\,(\frac{d^2}{d\theta^2}v(\theta))(2\,(\frac{d}{d\theta}P_\infty(\theta))^2+2\,\frac{d^2}{d\theta^2}P_\infty(\theta)+4\,e^{P_\infty(\theta)}\frac{d}{d\theta}\psi_Q(\theta) \tag{11}$$
$$+8\,e^{P_\infty(\theta)}\psi_Q(\theta)\frac{d}{d\theta}P_\infty(\theta)+\frac{d^2}{d\theta^2}v(\theta))t\ln(t)e^{\gamma(t,\theta)}$$

as $t\to 0$ so that with $e^{\gamma(t,\theta)}\sim 1/t$ then $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ diverges only like $\ln(t)$. In the special case

$$\frac{d^2}{d\theta^2}v\left(\theta\right)=-2\left(\frac{d}{d\theta}P_\infty\left(\theta\right)\right)^2-2\,\frac{d^2}{d\theta^2}P_\infty\left(\theta\right)-4\,e^{P_\infty(\theta)}\frac{d}{d\theta}\psi_Q\left(\theta\right)-8\,e^{P_\infty(\theta)}\psi_Q\left(\theta\right)\frac{d}{d\theta}P_\infty\left(\theta\right)\neq 0 \tag{12}$$

we need to specify $V_1(\theta)$ in (7) which simplifies to

$$V_1(\theta)=(e^{P_\infty(\theta)})^2(\psi_Q(\theta))^2+\frac{1}{4}\frac{d^2}{d\theta^2}P_\infty(\theta). \tag{13}$$

Now we find $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\to 0$ as $t\to 0$ as shown in limitfivespecialdegen.mws

VII. $v(\theta)=1,\ \frac{dv(\theta)}{d\theta}=\frac{d^2Q_\infty(\theta)}{d\theta^2}=\frac{d^2}{d\theta^2}v(\theta)=\frac{d^3}{d\theta^3}Q_\infty(\theta)=0$

If $\frac{d^2}{d\theta^2}v(\theta)=0$ and $\frac{d^3}{d\theta^3}Q_\infty(\theta)=0$ we need to specify $V_1(\theta)$ in (7) which again simplifies to (13). Now we find

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\to R(\theta)te^{\gamma(t,\theta)} \tag{14}$$

where

$$R(\theta)=3\,(\frac{d^2}{d\theta^2}P_\infty(\theta)+(\frac{d}{d\theta}P_\infty(\theta))^2-2\,(\frac{d}{d\theta}\psi_Q(\theta))e^{P_\infty\theta)}-4\,(\frac{d}{d\theta}P_\infty(\theta))e^{P_\infty(\theta)}\psi_Q(\theta)) \tag{15}$$
$$(\frac{d^2}{d\theta^2}P_\infty(\theta)+(\frac{d}{d\theta}P_\infty(\theta))^2+2\,(\frac{d}{d\theta}\psi_Q(\theta))e^{P_\infty(\theta)}+4\,(\frac{d}{d\theta}P_\infty(\theta))e^{P_\infty(\theta)}\psi_Q(\theta))$$

This is shown in limitsix.mws. Note that inclusion of $t^4Z(t,\theta)$ and $t^4W(t,\theta)$ does not change this result. See limitsixzw.mws. If $v(\theta)=1,\ \frac{dv(\theta)}{d\theta}=\frac{d^2Q_\infty(\theta)}{d\theta^2}=\frac{d^2}{d\theta^2}v(\theta)=\frac{d^3}{d\theta^3}Q_\infty(\theta)=0$ then with $e^{\gamma(t,\theta)}\sim 1/t$ it follows that $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ does not diverge as $t\to 0$. In the special case

$$\psi_Q\left(\theta\right)=\left(\int\left(\pm 1/2\,\frac{d^2}{d\theta^2}P_\infty\left(\theta\right)-1/2\,\left(\frac{d}{d\theta}P_\infty\left(\theta\right)\right)^2\right)e^{P_\infty(\theta)}d\theta+\_C1\right)e^{-2\,P_\infty(\theta)} \tag{16}$$

$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\to 0$ as $t\to 0$ as shown in limitseven.mws

VIII. $v(\theta)=-1,\ \frac{d^2Q_\infty(\theta)}{d\theta^2}\neq 0$

With (7) and (8) the consistency conditions are now (see constraintvminusone.mws)

$$\psi_Q(\theta)=\frac{d}{d\theta}\psi_Q(\theta)=0. \tag{17}$$

We now find

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\ \to 1/4\,\frac{(e^{P_\infty(\theta)})^6(\frac{d^2}{d\theta^2}Q_\infty\theta))^6(\ln(t))^6e^{\gamma(t,\theta)}}{t^9} \tag{18}$$

as $t\to 0$. For $v(\theta)=-1,\ \frac{d^2Q_\infty(\theta)}{d\theta^2}\neq 0$ with $e^{\gamma(t,\theta)}\sim 1/t$ then $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ diverges like $\ln(t)^6/t^{10}$. See limiteight.mws.

IX. $v(\theta)=-1,\frac{d^2Q_\infty(\theta)}{d\theta^2}=0$

With (7) and (8) and the consistency conditions (17) we now obtain the following limits:

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\ \to -\frac{1}{256}\,\frac{(e^{P_\infty(\theta)})^6(\frac{d^3}{d\theta^3}Q_\infty(\theta))^6(\ln(t))^6e^{\gamma(t,\theta)}}{t^3} \tag{19}$$

for $\frac{d^3Q_\infty(\theta)}{d\theta^3}\neq 0$, see limitnine.mws,

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\ \to\frac{(e^{P_\infty(\theta)})^2(\frac{d^4}{d\theta^4}Q_\infty(\theta))^2(\ln(t))^2e^{\gamma(t,\theta)}}{t} \tag{20}$$

for $\frac{d^4Q_\infty(\theta)}{d\theta^4}\neq 0$, see limitten.mws,

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\ \to\frac{\left(-4\,\left(\frac{d}{d\theta}v\left(\theta\right)\right)^2+4\,e^{2\,P_\infty(\theta)}\left(\frac{d^4}{d\theta^4}\psi_Q\left(\theta\right)\right)^2\right)e^{\gamma(t,\theta)}}{t} \tag{21}$$

for $((\frac{d}{d\theta}v(\theta))^2\neq e^{2\,P_\infty(\theta)}(\frac{d^4}{d\theta^4}\psi_Q(\theta))^2)e^{\gamma(t,\theta)}$, see limiteleven.mws. The cases $\frac{d}{d\theta}v(\theta)=0$ or $\frac{d^4}{d\theta^4}\psi_Q(\theta)=0$ follow from (21).

X. $v(\theta) = -1, \frac{d^2 Q_\infty(\theta)}{d\theta^2} = \frac{d^3 Q_\infty(\theta)}{d\theta^3} = \frac{d^4 Q_\infty(\theta)}{d\theta^4} = \frac{d}{d\theta} v(\theta) = \frac{d^4}{d\theta^4} \psi_Q(\theta) = 0$

With (7) and (8) and the consistency conditions (17) we now obtain the following limits:

$$C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} \;\; \rightarrow 4\, \frac{e^{2\, P_\infty(\theta)} (\frac{d^2}{d\theta^2} \psi_Q(\theta))^2 e^{\gamma(t,\theta)}}{t} \tag{22}$$

for $\frac{d^2}{d\theta^2} \psi_Q(\theta) \neq 0$ as in limittwelve.mws and

$$C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} \;\; \rightarrow 3\, t (\ln(t))^2 e^{\gamma(t,\theta)} (\frac{d^2}{d\theta^2} v(\theta))^2 \tag{23}$$

for $\frac{d^2}{d\theta^2} \psi_Q(\theta) = 0$ but $\frac{d^2}{d\theta^2} v(\theta) \neq 0$ as in limitthirteen.mws. If in addition $\frac{d^2}{d\theta^2} v(\theta) = 0$ we note that $V_1(\theta) = 1/4\, \frac{d^2}{d\theta^2} P_\infty(\theta)$ and we obtain

$$C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} \;\; \rightarrow 3\, t e^{\gamma(t,\theta)} ((\frac{d^2}{d\theta^2} P_\infty(\theta))^2 + (\frac{d}{d\theta} P_\infty(\theta))^4 - 4\, e^{2\, P_\infty(\theta)} (\frac{d}{d\theta} Q_\infty(\theta))^2 - 2\, (\frac{d^2}{d\theta^2} P_\infty(\theta)) (\frac{d}{d\theta} P_\infty(\theta))^2). \tag{24}$$

In the special case

$$P_\infty(\theta) = -\ln(\pm(-2\, \theta\, Q_\infty(\theta) + \_C1\, \theta + 2 \int \theta\, \frac{d}{d\theta} Q_\infty(\theta) d\theta - \_C2)) \tag{25}$$

$C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} \rightarrow 0$ as $t \rightarrow 0$. These cases are shown in limitfourteen.mws