

Demonstration 1 (kruskalo): Transformation to original Kruskal coordinates; reduction of Ricci to zero, calculation and simplification of the Kretschmann scalar in these new coordinates.

```
> restart;
> grtw();
```

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Developed by Peter Musgrave, Denis Pollney and Kayll Lake

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e:/Grtii(6)/Metrics

First we load in the original Schwarzschild metric

```
> qload(schw);
```

Default spacetime = schw

For the schw spacetime:

Coordinates

x(up)

x^a = [r, θ, φ, t]

Line element

$$ds^2 = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2 + \left(-1 + \frac{2m}{r}\right) dt^2$$

The Schwarzschild metric in curvature coordinates

The following transformations are given by Kruskal:

```
> xform := [u(r,t) = sqrt(r/(2*m)-1)*exp(r/(4*m))*cosh(t/(4*m)), v(r,t) = sqrt(r/(2*m)-1)*exp(r/(4*m))*sinh(t/(4*m)), Theta(theta) = theta, Phi(phi) = phi];
```

$$xform := \left[\begin{aligned} u(r,t) &= \frac{1}{2} \sqrt{2 \frac{r}{m} - 4} e^{\left(\frac{1}{4} \frac{r}{m}\right)} \cosh\left(\frac{1}{4} \frac{t}{m}\right), \\ v(r,t) &= \frac{1}{2} \sqrt{2 \frac{r}{m} - 4} e^{\left(\frac{1}{4} \frac{r}{m}\right)} \sinh\left(\frac{1}{4} \frac{t}{m}\right), \Theta(\theta) = \theta, \Phi(\phi) = \phi \end{aligned} \right]$$

We now transform the metric tensor:

```
> grtransform(schw,kruskalo,xform):
```

The new default metric is: kruskalo

We simplify and display the new form of the metric:

```
> gralter(g(dn,dn),1);
```

Component simplification of a GRTensorII object:

Applying routine simplify to object g(dn,dn)

CPU Time = .100

```
> grdisplay(_);
```

For the kruskalo spacetime:

Covariant metric tensor

$g(dn, dn)$

$$g_{ab} = \begin{bmatrix} 32 \frac{m^3 e^{\left(-1/2 \frac{r}{m}\right)}}{r} & 0 & 0 & 0 \\ 0 & -32 \frac{m^3 e^{\left(-1/2 \frac{r}{m}\right)}}{r} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & -r^2 (-1 + \cos(\theta)^2) \end{bmatrix}$$

Note that in the new coordinates $[u,v,\Theta,\Phi]$ $r=r(u,v)$. We finish the transformation as follows:

```
> grmap(g(dn,dn),subs,r=r(u,v),theta=Theta,phi=Phi,`x`);
```

Applying routine subs to g(dn,dn)

We force MapleV to use sine as follows:

```
> gralter(g(dn,dn),11);
```

Component simplification of a GRTensorII object:

Applying routine `simplify[trigsin]` to object g(dn,dn)

CPU Time = .010

```
> grdisplay(g(dn,dn));
```

For the kruskalo spacetime:

Covariant metric tensor

$g(dn, dn)$

$$g_{ab} = \begin{bmatrix} 32 \frac{m^3 e^{\left(-1/2 \frac{r(u,v)}{m}\right)}}{r(u,v)} & 0 & 0 & 0 \\ 0 & -32 \frac{m^3 e^{\left(-1/2 \frac{r(u,v)}{m}\right)}}{r(u,v)} & 0 & 0 \\ 0 & 0 & r(u,v)^2 & 0 \\ 0 & 0 & 0 & r(u,v)^2 \sin(\Theta)^2 \end{bmatrix}$$

This is the original Kruskal metric.

We now wish to actually do some calculations with it. Because of the implicit nature of the metric, it is subject to the constraints which define $r(u,v)$. We chose to eliminate the exponential function in what follows.

```
> P(u,v):=u^2-v^2-simplify((sqrt(r(u,v)/(2*m)-1)*exp(r(u,v)/(4*m))*cosh(t/(4*m)))^2-(sqrt(r(u,v)/(2*m)-1)*exp(r(u,v)/(4*m))*sinh(t/(4*m)))^2);
```

$$P(u, v) := u^2 - v^2 - \frac{1}{2} \frac{(r(u, v) - 2m) e^{\left(\frac{1}{2} \frac{r(u, v)}{m}\right)}}{m}$$

> `simplify(solve(P(u,v)=0,exp(r(u,v)/(2*m))));`

$$-2 \frac{m(-u^2 + v^2)}{r(u, v) - 2m}$$

> `P(u,v)=0;`

$$u^2 - v^2 - \frac{1}{2} \frac{(r(u, v) - 2m) e^{\left(\frac{1}{2} \frac{r(u, v)}{m}\right)}}{m} = 0$$

> `solve(diff(%,v),diff(r(u,v),v));`

$$-8 \frac{v m^2}{e^{\left(\frac{1}{2} \frac{r(u, v)}{m}\right)} r(u, v)}$$

> `B(u,v):=subs(exp(r(u,v)/(2*m))=2*(-u^2+v^2)*m/(-r(u,v)+2*m),%);`

$$B(u, v) := -4 \frac{v m (-r(u, v) + 2m)}{(-u^2 + v^2) r(u, v)}$$

> `P(u,v)=0;`

$$u^2 - v^2 - \frac{1}{2} \frac{(r(u, v) - 2m) e^{\left(\frac{1}{2} \frac{r(u, v)}{m}\right)}}{m} = 0$$

> `solve(diff(%,u),diff(r(u,v),u));`

$$8 \frac{u m^2}{e^{\left(\frac{1}{2} \frac{r(u, v)}{m}\right)} r(u, v)}$$

> `A(u,v):=subs(exp(r(u,v)/(2*m))=2*(-u^2+v^2)*m/(-r(u,v)+2*m),%);`

$$A(u, v) := 4 \frac{u m (-r(u, v) + 2m)}{(-u^2 + v^2) r(u, v)}$$

The functions A(u,v) and B(u,v) constrain the derivatives of r(u,v) wrt u and v. We now write the Kruskal metric without the exponent and attach these constraints.

> `grmap(g(dn,dn),subs(exp(-r(u,v)/(2*m))=(r(u,v)-2*m)/(2*m*(u^2-v^2)),`x`));`

Applying routine subs to g(dn,dn)

> `grdisplay(_);`

For the kruskalo spacetime:

Covariant metric tensor

g(dn, dn)

$$g_{ab} = \begin{bmatrix} 16 \frac{m^2 (r(u, v) - 2m)}{(u^2 - v^2) r(u, v)} & 0 & 0 & 0 \\ 0 & -16 \frac{m^2 (r(u, v) - 2m)}{(u^2 - v^2) r(u, v)} & 0 & 0 \\ 0 & 0 & r(u, v)^2 & 0 \\ 0 & 0 & 0 & r(u, v)^2 \sin^2(\Theta) \end{bmatrix}$$

```
> grconstraint(kruskalo);
```

```
Constraint specification and manipulation
```

```
Do you wish to
```

- 1) Add a constraint to the metric
- 2) Remove a constraint from the metric
- 3) Modify a metric constraint
- 4) Display the existing constraints
- 5) Exit

```
Enter 1-5 >
```

```
grconstraint>1;
```

```
1
```

```
grconstraint>diff(r(u,v),u)=A(u,v);
```

$$\frac{\partial}{\partial u} r(u, v) = 4 \frac{u m (-r(u, v) + 2m)}{(-u^2 + v^2) r(u, v)}$$

```
grconstraint>1;
```

```
1
```

```
grconstraint>diff(r(u,v),v)=B(u,v);
```

$$\frac{\partial}{\partial v} r(u, v) = -4 \frac{v m (-r(u, v) + 2m)}{(-u^2 + v^2) r(u, v)}$$

```
grconstraint>4;
```

```
4
```

```
grconstraint>5;
```

```
5
```

We now save the Kruskal metric with the constraints.

```
> grsaveg(kruskalo);
```

```
Information written to: `e:/Grtii(6)/Metrics/kruskalo.mpl`
```

Note that there is now NO reference to the original coordinates.

We go on now to show that the metric is vacuum, and calculate the Kretschmann scalar.

```
> grcalc(R(dn,dn));
```

```
CPU Time = .080
```

```
> gralter(_,13,7);
```

```
Component simplification of a GRTensorII object:
```

```
Applying routine `Apply constraints repeatedly` to object R(dn,dn)
```

```
Applying routine factor to object R(dn,dn)
```

```
CPU Time = .030
```

```
> grdisplay(_);
```

For the kruskalo spacetime:

Covariant Ricci

$R(dn, dn)$

$R_{a b} = \text{All components are zero}$

> **grcalc(RiemSq);**

Created definition for R(dn,dn,up,up)

CPU Time = .240

> **gralter(_,13,7);**

Component simplification of a GRTensorII object:

Applying routine `Apply constraints repeatedly` to object RiemSq

Applying routine factor to object RiemSq

CPU Time = .060

> **grdisplay(_);**

For the kruskalo spacetime:

Full Contraction of Riemann

$$K = 48 \frac{m^2}{r(u, v)^6}$$

>