Demonstration 1 (godel): An elementary study of the G\"{o}del metric.

```
> restart:
> grtw();
```

GRTensorII Version 1.79 (R6)

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Latest version available from: http://grtensor.phy.queensu.ca/

e:/Grtii(6)/Metrics

```
> grOptionTrace:=false:
> qload(godel1);
```

Default spacetime = godel1

For the godel1 spacetime:

Coordinates

$$x(up)$$

$$x^a = [t, x, y, z]$$

Line element

$$ds^2 = -d\,t^2 - 2\,e^{(\sqrt{2}\,\omega\,x)}\,d\,t\ d\,y + d\,x^2 - \frac{1}{2}\,e^{(2\sqrt{2}\,\omega\,x)}\,d\,y^2 + d\,z^2$$

The Godel universe (c.f. Hawking and Ellis Section 5.7)

First we define a velocity field and then examine the kinematics

```
> grdef(`u{^a}:=[1,0,0,0]`);
Components assigned for metric: godel1
Created definition for u(up)
> grcalc(acc[u](up),expsc[u],shear[u](up,up),vor[u]);
Created a definition for u(up,cdn)
Created definition for shear(up,up)
Created definition for u(dn)
Created a definition for u(dn,cdn)
Created definition for acc(dn)
Created definition for vor(up,dn)
```

CPU Time $= .419$

```
> gralter(_,radical,expand,factor);
Component simplification of a GRTensorII object:

Applying routine `simplify[radical]` to object acc(up)[u]
Applying routine `simplify[radical]` to object expsc[u]
Applying routine `simplify[radical]` to object shear(up,up)[u]
Applying routine `simplify[radical]` to object vor[u]
Applying routine expand to object acc(up)[u]
Applying routine expand to object expsc[u]
Applying routine expand to object shear(up,up)[u]
Applying routine expand to object vor[u]
Applying routine factor to object acc(up)[u]
Applying routine factor to object expsc[u]
Applying routine factor to object shear(up,up)[u]
```

```
Applying routine factor to object vor[u]
```

$$CPU\ Time\ = .080$$

```
> grdisplay(_);
```

For the godel1 spacetime:

Acceleration vector

$$\mathrm{acc}(up)$$

$$a^a = \textit{All components are zero}$$

Expansion scalar

$$\Theta = 0$$

shear(up,up)

$$\mathrm{shear}(up, up)$$

$$\sigma^{a\ b} = \textit{All components are zero}$$

Vorticity scalar

$$omega[u] = \sqrt{2}\sqrt{\omega^2}$$

Einstein tensor is augmented with λ (the cosmological constant).

```
> grdef(`En{a b}:=G{a b}+lambda*g{a b}`);
Created definition for En(dn,dn)
> grcalc(En(up,up));
Created definition for En(up,up)
```

$$CPU\ Time\ = .040$$

```
> gralter(_,expand,factor);
Component simplification of a GRTensorII object:

Applying routine expand to object En(up,up)
Applying routine factor to object En(up,up)
```

$$CPU\ Time\ = .010$$

```
> grdisplay(_);
```

For the godel1 spacetime:

En(up,up)

$$\mathrm{En}(up, up)$$

$$En^{a\ b} = \begin{bmatrix} 3\omega^2+\lambda & 0 & -2\dfrac{\omega^2+\lambda}{e^{(\sqrt{2}\,\omega\,x)}} & 0 \\ 0 & \omega^2+\lambda & 0 & 0 \\ -2\dfrac{\omega^2+\lambda}{e^{(\sqrt{2}\,\omega\,x)}} & 0 & 2\dfrac{\omega^2+\lambda}{(e^{(\sqrt{2}\,\omega\,x)})^2} & 0 \\ 0 & 0 & 0 & \omega^2+\lambda \end{bmatrix}$$

It is clear that for dust

```
> lambda:=-omega^2;
```

$$\lambda := -\omega^2$$

```
> grdisplay(_);
```

For the godel1 spacetime:

En(up,up)

En(*up*, *up*)

$$En^{a\ \ b} = \begin{bmatrix} 2\,\omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We now calculate the electric and magnetic components of the Weyl tensor associated with the velocity field u.

```
> grcalc(E[u](dn,dn),H[u](dn,dn));
```

$$CPU\ Time\ = .080$$

```
> gralter(_,6,7);
Component simplification of a GRTensorII object:

Applying routine expand to object E(dn,dn)[u]
Applying routine expand to object H(dn,dn)[u]
Applying routine factor to object E(dn,dn)[u]
Applying routine factor to object H(dn,dn)[u]
```

$$CPU\ Time\ = .020$$

```
> grdisplay(_);
```

For the godel1 spacetime:

Electric part of Weyl

E(*dn*, *dn*)

$$E_{a\ \ b} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3}\,\omega^2 & 0 & 0 \\ 0 & 0 & \frac{1}{6}\,\omega^2\,(e^{(\sqrt{2}\,\omega\,x)})^2 & 0 \\ 0 & 0 & 0 & -\frac{2}{3}\,\omega^2 \end{bmatrix}$$

Magnetic part of Weyl

H(*dn*, *dn*)

$H_{a\ \ b}$ = *All components are zero*

It is clear that the space has a high degree of symmetry. Here is a quick look for Killing vectors:

```
> KillingCoords();
Testing Killing coordinates for godel1
Created definition for coord1(dn)
Created a definition for coord1(dn,cdn)
Created a definition for coord1(up,cdn)
Created definition for coord2(dn)
Created a definition for coord2(dn,cdn)
Created a definition for coord2(up,cdn)
Created definition for coord3(dn)
```

```
Created a definition for coord3(dn,cdn)
Created a definition for coord3(up,cdn)
Created definition for coord4(dn)
Created a definition for coord4(dn,cdn)
Created a definition for coord4(up,cdn)
```

$$CPU\ Time\ = .842$$

Killing Coordinate Test Results

$$Coordinate\ vector\ = [t, x, y, z]$$

$$\mathbf{coord1}(up) = [1, 0, 0, 0],\ \ a\ Killing\ vector.$$

$$\mathbf{coord2}(up) = [0, 1, 0, 0],\ \ not\ a\ Killing\ vector.$$

$$\mathbf{coord3}(up) = [0, 0, 1, 0],\ \ a\ Killing\ vector.$$

$$\mathbf{coord4}(up) = [0, 0, 0, 1],\ \ a\ Killing\ vector.$$

>