Demonstration 2(em):Einstein-Maxwell equations in the Kerr-Newman spacetime.

```
> restart:
> grtw():
```

GRTensorII Version 1.79 (R6)

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```
> grOptionTermSize:=30:
> qload(newkn);
```

Default spacetime = newkn

For the newkn spacetime:

Coordinates

x(*up*)

$$x^a = [r, u, \phi, t]$$

Line element

$$ds^2 = \frac{(r^2+u^2)\ d\,r^2}{r^2 - 2\,m\,r + a^2 + Q^2} + \frac{(r^2+u^2)\ d\,u^2}{a^2-u^2} + \frac{(a^2-u^2)\left(r^2+a^2+\frac{(a^2-u^2)\,(2\,m\,r-Q^2)}{r^2+u^2}\right)\ d\,\phi^2}{a^2} - \frac{2\,(a^2-u^2)\,(2\,m\,r-Q^2)\ d\,\phi\ d\ t}{a\,(r^2+u^2)} + \left(-1+\frac{2\,m\,r-Q^2}{r^2+u^2}\right)\ d\,t^2$$

$$Constraints = [u = a\cos(\theta)]$$

*Kerr Newman Solution in Boyer-Lindquist coordinates (u=a*cos(theta))*

a)Background:

Define the vector potential A_{a} and the electromagnetic field tensor f_{ab}:

```
> grdef(`A{a}:=[0,0,Q*r*(a^2-u^2)/(a*(r^2+u^2)),-Q*r/(r^2+u^2)]`);
Components assigned for metric: newkn
Created definition for A(dn)
> grdisplay(A(dn));
```

For the newkn spacetime:

A(dn)

A(*dn*)

$$A_a = \left[0,\ 0,\ \frac{Q\,r\,(a^2-u^2)}{a\,(r^2+u^2)},\ -\frac{Q\,r}{r^2+u^2}\right]$$

```
> grdef(`f{[a b]}:=2*A{[b ; a]}`);
Created a definition for A(dn,cdn)
```

```
Created definition for f(dn,dn)
```

Two scalar invariants associated with the field tensor are \$f_{ab}f^{ab}\$ and \$f_{ab}f*^{ab}\$:

```
> grdef(`I1:=f{a b}*f{^a^b}`);
Created definition for f(up,up)
Created definition for I1
> grdef(`fstar{a b}:=LevC{a b c d}*f{^c^d}`);
Created definition for fstar(dn,dn)
> grdef(`I2:=f{a b}*fstar{^a^b}`);
Created definition for fstar(up,up)
Created definition for I2
```

The 4-current is defined by\$J^{a}=f^{ab}_{~~;b}/(4\pi)\$

```
> grdef(`J{^a}:=f{^a^b;b}/(4*Pi)`);
Created a definition for f(up,up,cdn)
Created definition for J(up)
```

and for the field tensor we construct the permutation \$f_{ab;c}+f_{bc;a}+f_{ca;b}\$.

```
> grdef(`Id{a b c}:=f{a b ; c}+f{b c ; a}+f{c a ; b}`);
Created a definition for f(dn,dn,cdn)
Created definition for Id(dn,dn,dn)
```

b)Calculations: First we calculate, simplify and display the field tensor and invariants. The four-current and permutaion of the field tensor are then reduced to zero.

```
> grcalc(f(dn,dn));
```

$$CPU\ Time\ = .080$$

```
> gralter(_,2,7);
Component simplification of a GRTensorII object:

Applying routine `simplify[trig]` to object f(dn,dn)
Applying routine factor to object f(dn,dn)
```

$$CPU\ Time\ = .030$$

```
> grdisplay(_);
```

For the newkn spacetime:

f(dn,dn)

$$\mathrm{f}(dn, dn)$$

$$f_{r\,\phi} = -\frac{(r-u)\,(r+u)\,Q\,(a-u)\,(a+u)}{a\,(r^2+u^2)^2}$$

$$f_{r\,t} = \frac{Q\,(r-u)\,(r+u)}{(r^2+u^2)^2}$$

$$f_{u\,\phi} = -2\,\frac{Q\,r\,u\,(a^2+r^2)}{a\,(r^2+u^2)^2}$$

$$f_{u\,t} = 2\,\frac{Q\,r\,u}{(r^2+u^2)^2}$$

```
> grcalc(I1,I2);
```

$$CPU\ Time\ = .060$$

```
> gralter(_,2,7,10);
Component simplification of a GRTensorII object:

Applying routine `simplify[trig]` to object I1
Applying routine `simplify[trig]` to object I2
Applying routine factor to object I1
Applying routine factor to object I2
Applying routine `simplify[sqrt]` to object I1
Applying routine `simplify[sqrt]` to object I2
```

$$CPU\ Time\ = .060$$

```
> grdisplay(_);
```

For the newkn spacetime:

I1

$$I1 = -2\,\frac{Q^2\,(-u^2+2\,r\,u+r^2)\,(-u^2-2\,r\,u+r^2)}{(r^2+u^2)^4}$$

I2

$$I2 = 16\,\frac{(r-u)\,(r+u)\,Q^2\,r\,u}{(r^2+u^2)^4}$$

```
> grcalc(J(up));
```

$$CPU\ Time\ = .050$$

```
> grdisplay(_);
```

For the newkn spacetime:

J(up)

J(*up*)

$$J^a = All\ components\ are\ zero$$

```
> grcalc(Id(dn,dn,dn));
```

$$CPU\ Time\ = .070$$

```
> grdisplay(_);
```

For the newkn spacetime:

Id(dn,dn,dn)

Id(dn,dn,dn) = All components are zero

c)Einstein-Maxwell equations: We construct the energy-momentum tensor out of the field tensor ($T_{ab}=(f_{ac}f_{b}^{~c}-g_{ab}f_{cd}f^{cd}/4)/(4\pi)$) and then show that the Kerr-Newman spacetime satifies the Einstein-Maxwell equations.

```
> grdef(`T{a b}:=(+f{a c}*f{b^c}-g{a b}*I1/4)/(4*Pi)`);
Created definition for f(dn,up)
Created definition for T(dn,dn)
> grdef(`Dif{a b}:=G{a b}-8*Pi*T{a b}`);
Created definition for Dif(dn,dn)
> grcalc(Dif(dn,dn));
```

$$CPU\ Time\ = .090$$

```
> gralter(_,6,7);
Component simplification of a GRTensorII object:

Applying routine expand to object Dif(dn,dn)
Applying routine factor to object Dif(dn,dn)
```

$$CPU\ Time\ = .010$$

```
> grdisplay(_);
```

For the newkn spacetime:

Dif(dn,dn)

$$\mathrm{Dif}(dn, dn)$$

$$Dif_{a\ b} = All\ components\ are\ zero$$

```
>
```